Avalanches in a cellular automaton

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(Received 10 March 1994; revised manuscript received 21 December 1994)

We study a cellular automaton that generates avalanches reminiscent of those occurring in granular materials and fault displacements. We show that the temporal evolution of a suitably defined macroscopic variable, the mean energy of the system, can be described as a stochastic Markov process. We solve the corresponding master equation and confirm the appropriateness of the theory by comparing with the results of the automaton.

PACS number(s): 05.40.+j

I. INTRODUCTION

Long periods of mechanical stability with sudden catastrophic events in between are a ubiquitous characteristic of macroscopic systems subject to steady "external" loads. Such is the behavior of geological faults [1–3], which, subject to the stresses of the steady drift of the tectonic plates, results in sudden, somewhat local, long displacements of the fault, namely, earthquakes; an event such as this releases the stress accumulated and, in some sense, rests the system to a more stable state. It is also the behavior of the surface of a pile of sand [4–9] or of any granular material, which after adding more and more grains to, it reaches angles at which the pile is no longer stable and an avalanche occurs. By keeping the external agent acting at a steady rate, this behavior is repeated at irregular intervals.

In this article we study a two-dimensional cellular automation that shows the main features of the behavior described above. The model considered here is part of a wider class of models discussed in detail by Lomnitz-Adler, Knopoff, and Martinez-Mekler [10] and certainly inspired by the pioneering work of Bak, Tang, and Wiesenfeld [11]. The automaton under consideration, being externally "loaded" and due to its particular rules of evolution, displays the propagation of a local scalar quantity h_{ij}, here defined as energy, in a form reminiscent of an avalanche in a sandpile or a fracture in a geological fault. The macroscopic quantity under study is the total energy $H(t)$ of the system

$$H(t)=\frac{1}{N^2}\sum_{ij} h_{ij}(t) , \tag{1.1}$$

where the sum is over all the N^2 sites of the system.

Figure 1 shows a typical behavior of the variable $H(t)$: It is an *irregular* "saw-tooth" type of evolution with long time intervals during which the system is loaded at constant rates, followed by sudden "avalanches," which in a very short time drastically reduce the energy stored in the system. It is important to stress, and in fact is at the core of the present work, that *there is no* regularity whatsoever in the occurrence of avalanches, such as quasiperiodicity: see Fig. 2. The type of behavior shown by the "macroscopic" variable $H(t)$ appears strikingly similar to the corresponding variables in the occurrence of avalanches [5,8,9] in sandpiles and of displacements of geological faults [1,3]. We shall comment later on the possible relevance of the present work in the study of those problems.

Regarding the nonperiodicity of the behavior of $H(t)$, the main result we want to advance here is that the macroscopic quantity $H(t)$ is a Markovian stochastic process. Hence we can immediately write down a master equation for the probability distribution $P(H,t)$ to find the system with total energy H at time t. Since the avalanches all leave the system with essentially zero energy, we are able to exactly solve the master equation, albeit in terms of an unknown function. That is, the energy-dependent proba-

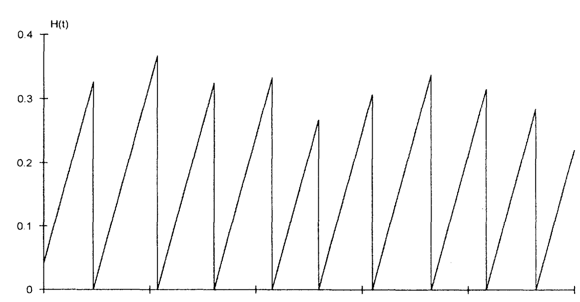

FIG. 1. Typical automaton output showing the mean energy per site H as a function of time steps.

*Deceased.

bility rate of occurrence of avalanches is, at the moment, unknown. However, distributions that depend on such a function can be independently measured: the energy distribution, the distribution of energies at which avalanches occur, the distribution of time intervals between successive avalanches, and, very important, the time evolution of the probability distribution $P(H,t)$. The theory predicts several relationships among these distributions and agreement with the automaton results is, as we shall see, very good. We argue that this is an appropriate way of describing the behavior of the pertinent macroscopic variable.

In a previous study by the present authors [8], it was shown that the angle $\alpha(t)$ of a sandpile in a rotating cylinder can also be described as a Markovian process and its description was formulated in terms of a similar master equation. Although the purpose now is the study *per se* of the stochastic dynamics of the automaton, we believe that we may also gain further understanding on the physical processes underlying real avalanches. That is, it is our viewpoint that the observed irregularity of a macroscopic variable, such as the energy of the automaton, the displacement of a fault, or the angle of a sandpile, is of a stochastic nature and can be very well described in terms of a master equation.

The stochastic behavior, we believe, is a consequence of the fact that for a given value of the macroscopic variable, there correspond *many* different microscopic stable configurations: Let us suppose we prepare an ensemble of systems all having the same value of the macroscopic variable, but in different microscopic states. The subsequent modification of those states by the slow external load eventually takes the system from those stable configurations to states that become unstable, but at a different value of the macroscopic variable. The highly dissipative, nonlinear, and complex dynamics of the underlaying microscopic degrees of freedom makes a description, at that level, a certainly difficult task. Nevertheless, there are many efforts along these directions [6,12–17] and as we shall argue in Sec. V, there must be a bridge, a "kineticlike theory," to connect both types of approaches.

The article is organized as follows. In Sec. II the cellular automaton is described in detail and the main numerical results are presented and discussed. Section III is devoted to the stochastic dynamics, the corresponding master equation, and its solution. In Sec. IV we compare the predictions of the theory with the results of the automaton. We make some final remarks in Sec. V.

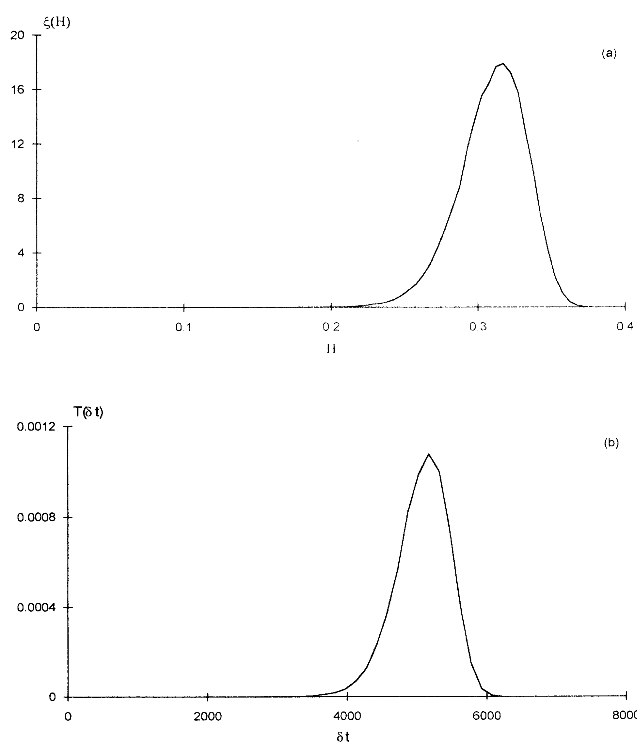

FIG. 2. (a) Distribution of initiation energies $\xi(H)$ vs H. (b) Distribution of waiting times $T(\delta t)$ vs δt.

II. THE CELLULAR AUTOMATON

The system is a two-dimensional square lattice, of finite size $N\times N$, in which at each site (i,j) a scalar variable h_{ij}, called energy here, is defined. The global variable that we follow is the mean energy per site, given by

$$H=\frac{1}{N^2}\sum_{ij} h_{ij} , \tag{2.1}$$

where the sum is over all the sites in the lattice.

At every discrete time step a fixed amount of energy ϵ is added to a randomly chosen site. When the energy of a site reaches or exceeds a prescribed threshold value h_c we say that the site "breaks": It transfers all of its energy in equal parts to its "unbroken" nearest neighbors and its energy is thus set equal to zero. Then, if the energy of an unbroken neighbor reaches or exceeds the threshold value, by virtue of receiving energy from the broken one, it also breaks and in turn transfers all of its energy in equal parts to its unbroken nearest neighbors. This process is repeated until no site involved in the rupture reaches or exceeds the threshold value. This "avalanche" is instantaneous in the sense that it occurs during one time step.

One of the main differences from similar models [11,18,19] is that while the avalanche or rupture is in progress, a broken site remains as such, i.e., it does not receive further energy from subsequent broken sites. Thus it may happen that during an avalanche a site finds itself with energy above the threshold, but with no unbroken neighbors; then the site loses all of its energy and it is set equal to zero. This is one of the dissipative mechanisms of the system. The other way in which the system loses energy is when a border site breaks: In this case, the border site transfers its energy in equal parts to its unbroken neighbors, including those hypothetical ones outside of the system. Namely, for the sites at the edges of the systems there is always at least *one* unbroken neighbor, while for the sites at the corners there are at least *two*. Thus, once an avalanche reaches the borders of the system, energy is lost.

For the particular simulation here presented we chose the size of the lattice $N=64$, the threshold energy value $h_c=1$, and the small amount added at each time step $\epsilon=0.25$. As already mentioned, this model shows a rich variety of phenomena that can be explored by changing the values of the different parameters and by considering different rules of microscopic evolution; part of this has already been discussed in Ref. [10]. Here we are interested in the similarities between the present model and the real physical systems we have mentioned. The relevant aspects of the process are qualitatively unchanged by considering larger systems. The output of the automaton is the time series of the mean energy $H(t)$; see Fig. 1. The results discussed below are all based on a run of approximately 7.5×10^8 time steps that involves approximately 10^5 large avalanches, namely, avalanches where the final state of the system has vanishing energy ($H\approx0$).

Our interest here resides in the large avalanches that involve the whole system, leaving it with practically no energy. This does not mean that there are no small avalanches capable of lowering the energy of the system; however, we can explicitly not consider them by introducing a small change in the rate of loading. In the following, we explain this approximation.

As pointed out, once the energy of a site reaches the threshold value h_c an avalanche begins. However, and very important, it turns out that the avalanches either are *very large,* meaning that all sites or all but less than 1% of the sites, break, or are *very small,* meaning that they involve a fraction of sites much smaller than the size of the system. Quantitatively, this means that the probability of occurrence of an avalanche decreases with its size up to a certain "critical size," above which the most probable avalanche is of the size of the system. This can be seen in Fig. 3, where we show the avalanche size distribution as a function of number of sites involved, for a run of 7.5×10^8 time steps: For avalanche sizes ranging from 1 to about 100 sites (i.e., less than or equal to 2% the size of the system) the distribution follows a characteristic power-law behavior (with exponent -3.5) common to these systems [10,9,12]. From about 100 to 4080 there are about 15 avalanches; that is, the probability of occurrence in this range is practically zero. From 4080 to the size of the system $N^2=4096$ there is again a large number of avalanches. Notice that the occurrence of avalanches of sizes N^2-1, N^2-2, etc. is at least two orders of magnitude less than the occurrence of avalanches of exactly the size of the system, N^2. It is also clear that the size distribution reaches a bimodal type of distribution and, for the number of time steps used here, we have verified that the distribution has reached its values for the small and the large avalanches. For the intermediate sizes the distribution is still noisy, but the probability of occurrence of those events is certainly smaller than 10^{-7}. It is of interest to mention here that recent experiments in avalanches, generated by adding grain after grain to sandpiles [9], have shown this clear-cut separation between small and large avalanches.

Now, if the avalanche is *large* as described above, the systems ends with practically zero energy $H(t)\approx0$. But, if the avalanche is *small,* then the total energy $H(t)$ does not necessarily change; it would do so only if the avalanche reaches the border or if a site finds itself with no unbroken neighbors. It turns out that between two consecutive large avalanches there are of the order of 200 small avalanches, and of those only 10% lower the energy of the system. However, the amount of energy released by a small avalanche is negligible to the extent that the quantity $H(t)$ is *insensitive* to them: Fig. 1 shows a typical behavior of $H(t)$ *including* both the small avalanches and the large avalanches with sizes greater than 98% of the size of the system. In the scale used the small ones are barely registered and all the large ones look as if they were the size of the system.

The above discussion explains the behavior of $H(t)$ as seen in Fig. 1 and it allows us to make a "coarse graining" approximation in order to consider only the "catastrophic" events in which all the energy stored in the system is lost. The way we neglect the small avalanches and

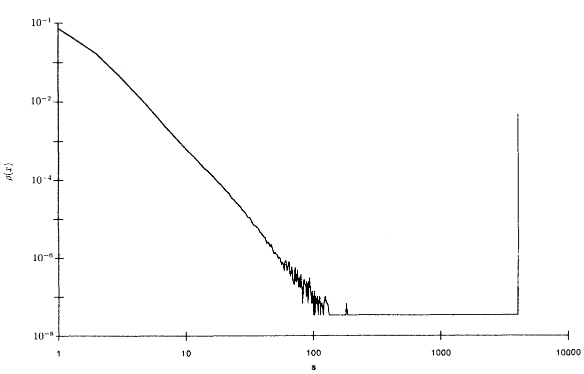

FIG. 3. Avalanche size distribution $\rho(x)$ as a function of the number of broken sites for 7.5×10^8 time steps.

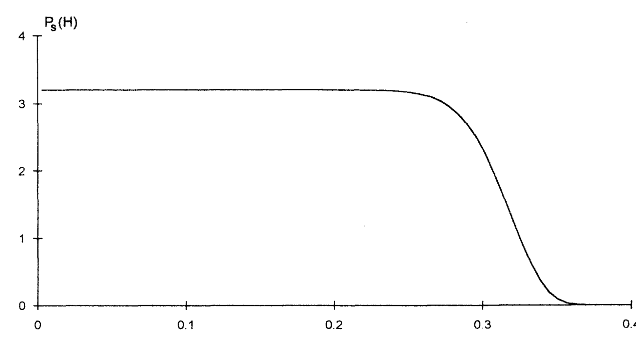

FIG. 4. Stationary probability distribution $P_s(H)$.

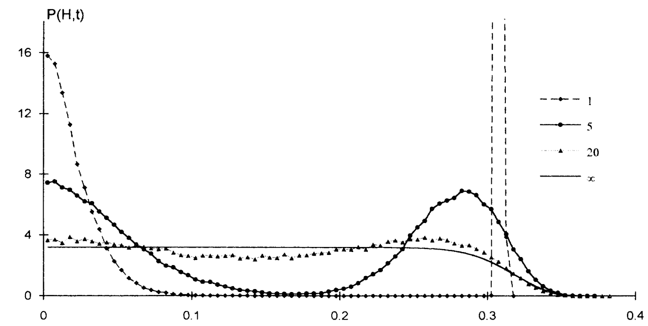

FIG. 5. Time-dependent distribution $P(H,t|0)$ vs t for several times $t=1$, 5, 20, and ∞. The latter is the stationary distribution of Fig. 4. The time t is in units of the average time $\langle \delta t \rangle$ between consecutive avalanches.

approximate the large ones that do not leave the system with exactly zero energy is by interpolating from $H=0$ to the value H at which the next large avalanche starts. By doing this, we necessarily incur in an error because the value of the "effective" loading rate Ω may change from avalanche to avalanche. This change, however, is less than 2% and it does not add another source of noise to be explicitly considered. Here we find the value $\Omega=(6.1\pm0.1)\times10^{-5}$; the uncertainty reflects the maximum deviation from the theoretical value $\Omega=\epsilon/(N^2\tau)$, where τ is the time step. This variation is already within the accuracy of the simulation. We want to point out that the coarse graining that we are doing does not mean that the small avalanches do not play any role; as we shall further discuss in Secs. III and V, their presence is included in the transition probability rate of occurrence of large avalanches. For definiteness, we shall refer to the latter as "avalanches" with no "large" qualifier. In Sec. III we present a theoretical framework to describe the coarse grained sequence $H(t)$ and in Sec. IV we compare predictions of the theory with distributions obtained from this sequence. The agreement shown then supports the statement that the variable $H(t)$ is well described as a stochastic Markov process. It should be borne in mind that any stochastic description [20] is always an approximation; the question is how good it is.

To end this section, we introduce the relevant distributions that can be independently extracted from the time series $H(t)$: (i) the (stationary) distribution $P_s(H)$, which gives the probability to find the system with a value of the energy between H and $H+dH$ (see Fig. 4); (ii) the distribution of energies $\xi(H)$ at which avalanches initiate [see Fig. 2(a)]; (iii) the distribution of time intervals between successive avalanches $T(\delta t)$ [see Fig. 2(b)]; and (iv) the distribution $P(H,t;0)$, which gives the probability to find the system with energy H at time t, given that at time $t=0$ the system had energy $H=0$ (see Fig. 5).

III. STOCHASTIC DYNAMICS

As introduced in the preceding section, the experimental output of the automaton is the mean energy per site as a function of time

$$H(t)=\frac{1}{N^2}\sum_{i,j} h_{ij}(t) , \tag{3.1}$$

where $h_{ij}(t)$ is the energy at site (i,j) and there are N^2 sites. The proposition here is that $H(t)$ is a Markovian stochastic process. Although the Markovian character is not evident, we shall develop the corresponding theory and, in the following sections, show that the predictions are in good agreement with the results of the simulation.

Let $P(H,t;H_0)$ be the conditional probability distribution to find the system with energy H at time t, given that at time $t=0$ the system had energy H_0. The Markovian property implies that $P(H,t;H_0)$ obeys a master equation [1,20,21]

$$\frac{\partial}{\partial t}P(H,t)+\Omega\frac{\partial}{\partial H}P(H,t) = \int_{-\infty}^{\infty} dH'[W(H,H')P(H',t)-W(H',H),P(H,t)] , \tag{3.2}$$

where $W(H,H')$ is the transition probability per unit of time that an avalanche will take the system from H' to H. This function bears all the relevant information of the physics of the avalanche initiation and propagation and it must be externally supplied. In our case, and in principle, it should be derivable from the "microscopic" dynamics of the automaton (in particular, it contains information about the small avalanches). At the moment, this function is unknown to us and its derivation is one of the future challenges of the present approach, but with its use in the master equation we are able to derive relationships between different and independently measured distribution functions and to verify the validity of this theory.

Let $Q_0(H)$ be the probability per unit of time for an avalanche to start at H regardless of where it ends. It is given by

$$Q_0(H)=\int_{-\infty}^{\infty} dH'W(H',H) . \tag{3.3}$$

Since all avalanches end with $H=0$, it follows that $W(H,H')$ must have the form

$$W(H,H')=\delta(H)Q_0(H') , \tag{3.4}$$

with $Q_0(H)=0$ for $H\le0$. Equation (3.2) takes the form

$$\frac{\partial}{\partial t}P(H,t)+\Omega\frac{\partial}{\partial H}P(H,t)+Q_0(H)P(H,t) =\delta(H)\int_0^{\infty} dH'Q_0(H')P(H',t) . \tag{3.5}$$

Let us first construct the *stationary* solution $P_s(H)$ of the master equation (3.5), which must be approached as $t\to\infty$. Moreover, since the process is stationary, $P_s(H)$ is also the one-time probability distribution [20]. $P_s(H)$ satisfies

$$\Omega\frac{\partial}{\partial H}P_s(H)+Q_0(H)P_s(H)=\delta(H)\int_0^{\infty} dH'Q_0(H')P_s(H') . \tag{3.6}$$

This is an integro-differential equation, but because of the form of the right-hand side it can be solved as a first-order differential equation with the right-hand side as a constant. By direct substitution it can be verified that the solution is

$$P_s(H)=\Theta(H)P_0e^{-I(H)} , \tag{3.7}$$

where $\Theta(H)$ is the Heavside function, $I(H)$ is given by

$$I(H)=\frac{1}{\Omega}\int_0^H dH' Q_0(H') , \tag{3.8}$$

and P_0 is the value of $P_s(H)$ at $H=0$. By virtue of the normalization condition, P_0 is also given by

$$P_0=\frac{1}{\int_0^\infty dH\, e^{-I(H)}} . \tag{3.9}$$

An important consequence can be readily found. First, integration of Eq. (3.6) over H, from $-\infty$ to ∞ yields $P_s(-\infty)=P_s(\infty)$; but from the solution Eq. (3.7) this implies that $P_s(-\infty)=P_s(\infty)=0$. Hence, for $H\to\infty$, $I(H)\to\infty$. However, from the way the model is defined, H cannot grow without bound and the divergence of $I(H)$ must occur at a finite $H_{\max}\le 1$. Since $Q_0(H)$ is a probability rate, we find that $Q_0(H)$ must also diverge as $H\to H_{\max}$ (for convenience, we shall keep ∞ as the upper limit of H). We note that $H_{\max}\le 0.75$ and, in principle, the equal sign should hold. From the simulation, however, the highest value that the energy seems to reach is at most $H_{\max}\approx 0.4$; we shall return to this point later on.

We can construct $\xi(H)$, the distribution of energies H at which avalanches *initiate,* as the probability for an avalanche to occur at H times the probability for the system to be at H, namely,

$$\xi(H)=\frac{Q_0(H)P_s(H)}{\int_0^\infty dH' Q_0(H')P_s(H')} =\frac{1}{\Omega}Q_0(H)e^{-I(H)} , \tag{3.10}$$

where the second line follows from Eqs. (3.7) and (3.8). Now, from this identification and the stationary master equation Eq. (3.6) it follows that

$$P_s(H)=\frac{c_1}{\Omega}\left[1-\int_0^H dH'\xi(H')\right] , \tag{3.11}$$

where c_1 is given by

$$c_1=\int_0^\infty dH Q_0(H)P_s(H) ; \tag{3.12}$$

this quantity is also the inverse of the average time between successive avalanches and in a given time series it is simply given by the total number of avalanches divided by the total "observation" time. This indicates that all the quantities involved in Eq. (3.11) can be independently measured; in Sec. IV we shall see that Eq. (3.11) is very accurately satisfied. It is interesting to note that we need only $P_s(H)$ in order to find $\xi(H)$; this is due to the *stationarity of the process;* see Ref. [20].

We now return to the construction of the solution of the time-dependent master equation (3.5). First we introduce the Laplace transform of $f(t)$ as

$$\tilde{f}(z)=\int_0^\infty dz\, e^{-zt}f(t) . \tag{3.13}$$

Hence, by Laplace transforming Eq. (3.5) we obtain an equation for $\tilde{P}(H,z)$, the Laplace transform of $P(H,t)$,

$$z\tilde{P}(H,z)-P(H,0)+\Omega\partial_H\tilde{P}(H,z)+Q_0(H)\tilde{P}(H,z) =\delta(H)\int_0^\infty dH' Q_0(H')\tilde{P}(H',z) , \tag{3.14}$$

where $P(H,0)$ is the probability distribution at *time* $t=0$, given by

$$P(H,0)=\delta(H-H_0) , \tag{3.15}$$

with H_0 an arbitrary value ($0\le H_0\le H_{\max}$). Note that the evolution of an arbitrary initial distribution $\mathcal{P}(H)$ [different from Eq. (3.15)] can be found from

$$\mathcal{P}(H,t)=\int dH_0 P(H,t|H_0)\mathcal{P}(H_0) . \tag{3.16}$$

The structure of Eq. (3.14) is very similar to that of the stationary equation (3.6) and can be solved as a first-order differential equation as well. By direct substitution it can be verified that its solution is

$$\tilde{P}(H,z)=\frac{1}{\Omega}e^{-I(H)-(H/\Omega)z}\{\Theta(H-H_0)e^{I(H_0)+(H_0/\Omega)z} +\Theta(H)[\mathcal{A}(H_0,z)-1]\} , \tag{3.17}$$

where $\mathcal{A}(H_0,z)$ is any function of its arguments; in our problem it is fixed by the normalization condition

$$\int_0^\infty dH\, \tilde{P}(H,z)=\frac{1}{z} . \tag{3.18}$$

For the sake of clarity and also to gain physical insight into the solution, let us specialize to the case in which $H_0=0$, i.e., $P(H,0)=\delta(H)$. That is, initially, the distribution represents an ensemble of systems all being prepared after an avalanche has just occurred. In this case, $\mathcal{A}(H_0,z)$ has a simple form and the solution is

$$\tilde{P}(H,z)=\Theta(H)\frac{1}{\Omega}e^{-I(H)-(H/\Omega)z}[1+\tilde{K}(z)] , \tag{3.19}$$

where the function $\tilde{K}(z)$ is given by

$$\tilde{K}(z)=\frac{\Omega}{z\int_0^\infty dH' e^{-I(H')-(H'/\Omega)z}}-1 . \tag{3.20}$$

Formally, the solution $P(H,t;0)$ is found by Laplace inverting $\tilde{P}(H,z)$, Eq. (3.19),

$$P(H,t;0)=\Theta(H)\frac{1}{\Omega}e^{-I(H)}\left[\delta\left[t-\frac{H}{\Omega}\right]+K\left[t-\frac{H}{\Omega}\right]\right] . \tag{3.21}$$

Several points are worth emphasizing: (a) It can be shown that $K(x)=0$ for $x\le 0$ and therefore $P(H,0;0)=\delta(H)$; (b) for $t>H_{\max}/\Omega$ the first term no longer contributes; and (c) $K(t-H/\Omega)\to P_0\Omega$ as $t\to\infty$, recovering the stationary solution.

Thus, in principle, the solution for all times can be constructed solely from the knowledge of $Q_0(H)$. We shall show below that indeed this theory describes the evolution of $H(t)$ as given by the automaton. However, we can gain further understanding by analyzing the structure of the function $\tilde{K}(z)$, which is intimately related to the distributions $\xi(H)$ and $T(\delta t)$.

First, we note that

$$\frac{z}{\Omega}\int_0^\infty dH\, e^{-I(H)-(H/\Omega)z} = -\int_0^\infty dH\, e^{-I(H)}\frac{d}{dH}e^{-(H/\Omega)z} = 1-\int_0^\infty dH\, e^{-(H/\Omega)z}\xi(H) , \tag{3.22}$$

where we have integrated by parts and used Eq. (3.10). Now, even though $\xi(H)$ *is not a function of time,* the above expression is formally its Laplace transform, namely, we can define

$$\tilde{\xi}(z/\Omega) = \int_0^\infty dH\, e^{-(H/\Omega)z}\xi(H) . \tag{3.23}$$

Therefore, by substitution of Eqs. (3.22) and (3.23) into Eq. (3.20) we obtain

$$\tilde{K}(z) = \frac{\tilde{\xi}(z/\Omega)}{1-\tilde{\xi}(z/\Omega)} , \tag{3.24}$$

that is, $K(t)$ is the "infinite convolution" of $\xi(H)$,

$$K(t) = \Omega\xi(\Omega t) + \Omega\int_0^{\Omega t} dH\, \xi(\Omega t - H)\xi(H) + \Omega\int_0^{\Omega t} dH\int_0^H dH'\xi(\Omega t - H)\xi(H-H')\xi(H') + \cdots . \tag{3.25}$$

Let us now introduce a related function $S(t)$: The probability (per unit of time) that in the interval of time between t and $t+dt$ an avalanche occurs, given that at time $t=0$ one has just taken place, *regardless* of how many avalanches occurred in between. This is formally given by

$$S(t) = \int_0^\infty dH\, Q_0(H)P(H,t;0) , \tag{3.26}$$

where $P(H,t;0)$ is the solution Eq. (3.21). By calculating the Laplace transform of $S(t)$ we get

$$\tilde{S}(z) = \int_0^\infty dH\, Q_0(H)\tilde{P}(H,z) , \tag{3.27}$$

which by virtue of Eqs. (3.19), (3.24), and (3.23) becomes

$$\tilde{S}(z) = \frac{\tilde{\xi}(z/\Omega)}{1-\tilde{\xi}(z/\Omega)} , \tag{3.28}$$

namely, $\tilde{S}(z)=\tilde{K}(z)$: The function $K(t)$ is the probability (per unit time) that at time t an avalanche occurs.

A further relationship is also worth obtaining. Because the rate of loading is constant, it follows that the value of H for a time t after an avalanche has occurred is given by

$$H = \Omega t . \tag{3.29}$$

Hence the distribution $\xi(H)$ of energies at which avalanches initiate is related to the distribution $T(\delta t)$ of waiting times δt between consecutive avalanches, as

$$T(\delta t) = \Omega\xi(\Omega\delta t) . \tag{3.30}$$

This also implies that $S(t)$ and $T(t)$ are related as (in z space)

$$\tilde{T}(z) = \frac{\tilde{S}(z)}{1+\tilde{S}(z)} . \tag{3.31}$$

These two functions are of interest, and of particular relevance in terms of their physical content, since they refer to the probability of occurrence of the events under consideration *without* referring to the details of the process (such as the precise state of the system at a given time); namely, they just refer to the *time* statistics of the occurrence of the events and as such, these distributions are susceptible to being measured. We shall return to this in Sec. V.

IV. COMPARISON BETWEEN THEORY AND THE AUTOMATON

In this section we compare some of the predictions of the theory just introduced with the numerical results of the cellular automaton. As we have mentioned, the output of the automaton is the (coarse grained) times series $H(t)$ (see Fig. 1) from which the distributions $P_s(H)$, $\xi(H)$, and $P(H,t|0)$ can be *independently* obtained. Also, we can measure the constant c_1 [cf. Eq. (3.12)] as the total number of avalanches divided by the total observation time. The theory predicts relationships among the above distributions that we now describe.

The first comparison concerns Eq. (3.11). As mentioned above, all the quantities involved in this equation can be independently measured. In Fig. 6 we show such a comparison: the left-hand side and the right-hand side of Eq. (3.11) are plotted in the same figure. Noting that there are no adjustable parameters, the agreement is remarkable.

The above relationship [Eq. (3.11)] follows rigorously from the master equation. However, it could also be pre-

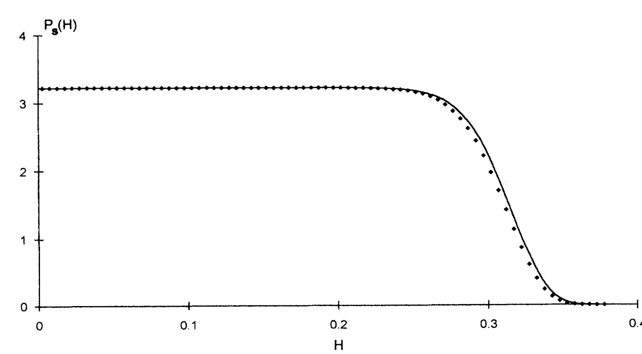

FIG. 6. Comparison of stationary distribution: as calculated from Eq. (3.11) (continuous line) and obtained directly from the automaton (dots).

dicted [22] without using the master equation, by *assuming* the avalanches to be *independent* events, which in this case turns out to be true due to the particular form of the transition probability distribution $W(H,H')$; see Eq. (3.4). In general, as is the case of avalanches in real sandpiles [8], for instance, the function $W(H,H')$ is not exactly factorable and therefore the avalanches are not independent events. One should be careful not to confuse "dependent events" with "memory in the process"; namely, the latter is a non-Markovian property while the former implies that there is a correlation between events at different times. In other words, *independence* is a stronger condition than *Markovian* [20].

A more stringent test for the master equation is the validity of its solution $P(H,t|0)$ [Eq. (3.21)] for arbitrary times $t>0$. This distribution can be directly extracted from a given times series $H(t)$. On the other hand, in order to calculate it from the theoretical expression Eq. (3.21) we would have to know $Q_0(H)$, the transition probability rate for an avalanche to start at H. The function $Q_0(H)$ can, in principle, be found from the knowledge of $P_s(H)$ only; see Eqs. (3.7) and (3.8). However, in order to calculate $P(H,t|0)$ by this route, a numerically accurate knowledge of $Q_0(H)$ would be necessary and, as we shall discuss below, this turns out to be very difficult to obtain. Instead, we present an argument of consistency of the theory and the simulation.

We found that the conditional probability distribution $P(H,t|0)$ is

$$P(H,t;0)=\Theta(H)\frac{1}{\Omega}e^{-I(H)}\left[\delta\left[t-\frac{H}{\Omega}\right]+K\left[t-\frac{H}{\Omega}\right]\right], \tag{4.1}$$

with $P(H,0|0)=\delta(H)$. Now, this distribution is actually a density, namely, it gives the probability in the interval H and $H+dH$. Hence the distribution in the interval $H=0$ and dH as a function of time is

$$P(0,t|0)=\Theta(H)\frac{1}{\Omega}[\delta(t)+K(t)], \tag{4.2}$$

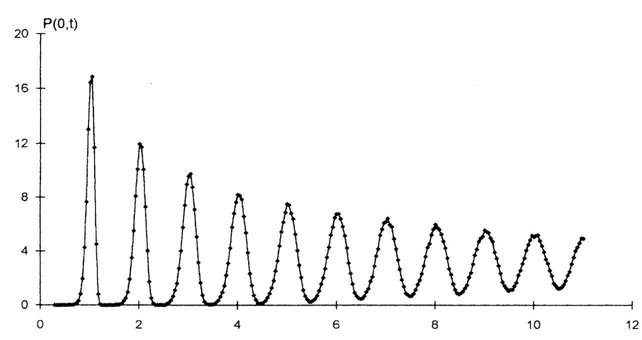

FIG. 7. Time-dependent distribution for $H=0$ and $P(0,t|0)$ as a function of time t, in units of the average time $\langle\delta t\rangle$ between consecutive avalanche.

and therefore the distribution for $H\neq 0$ can be written as

$$P(H,t|0)=e^{-I(H)}P\left[0,t-\frac{H}{\Omega}\middle|0\right]. \tag{4.3}$$

Moreover, from the stationary solution Eq. (3.7) we have

$$\frac{P_s(H)}{P_0}=e^{-I(H)}, \tag{4.4}$$

so that

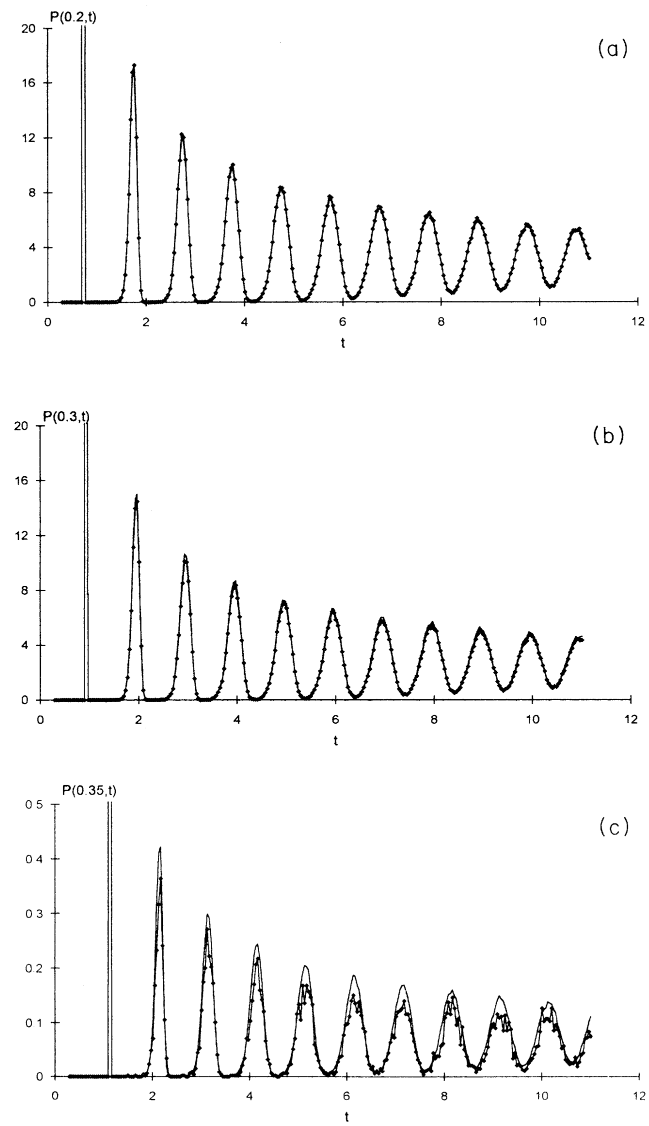

FIG. 8. Comparison of the time-dependent distribution $P(H,t|0)$ as a function of time: as calculated from Eq. (4.5) (continuous line) and obtained directly from the automaton (dots), for different values of the energy: (a) $H=0.2$, (b) $H=0.3$, (c) $H=0.35$. Note that the agreement deteriorates for large values of H; this is due to the poor statistics for those values (see the text). The time t is in units of the average time $\langle\delta t\rangle$ between consecutive avalanches.

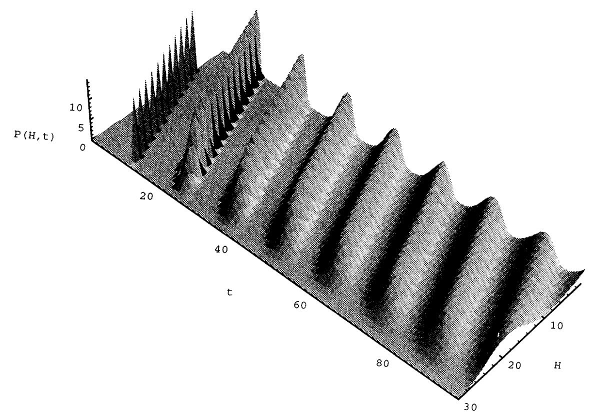

FIG. 9. Combined plot of $P(H,t|0)$ vs t and H.

$$P(H,t|0)=\frac{P_s(H)}{P_0}P\left[0,t-\frac{H}{\Omega}\middle|0\right]. \quad (4.5)$$

Hence we can *independently* extract $P(0,t|0)$, $P_s(H)$, P_0 and $P(H,t|0)$ and substitute them into Eq. (4.5) and compare. Figure 7 shows $P(0,t|0)$, Fig. 8 shows the comparison of the left- and the right-hand side of Eq. (4.5), and Fig. 9 shows a combined plot of $P(H,t|0)$ as a function of H and t. Again, there are no adjustable parameters and the agreement is very good. It is interesting to find that the behavior of $P(H,t|0)$ for arbitrary H and t is given by the knowledge of its asymptotic behavior for *large* times and *small* energies. The results shown indicate that the process is accurately described by the master equation.

V. FINAL REMARKS

We have studied a cellular automaton that, under the action of an external load, shows an irregular sequence of avalanches. By concentrating on the behavior of the energy $H(t)$ of the automaton, a global macroscopic variable, and not explicitly considering the dynamics of the microscopic degrees of freedom, we have been able to develop a theory that shows that in the appropriate time scale such a variable is well described as a stochastic Markovian process. As mentioned before, the stochasticity is not surprising since for any given value of H there correspond many microscopic configurations resulting in a type of intrinsic noise, common to many-body systems. However, the Markovian character of the process is not necessarily expected but, as we have shown, this appears to be case. This is the main result of the present article.

We would like to point out that the analysis of these automata and its stochastic treatment may be useful in the study of real systems. This type of idea has been already suggested to be relevant in the description of earthquakes [1,10]. Also, experiments conducted in rotating cylinders half filled with granular material [4,5,8] and in sandpiles grown by adding grain after grain [9] have shown that the angle of the granular surface behaves similarly to the energy of the automaton. Moreover, it has been shown [8] that the distribution of angles obeys the same type of master equation as that found here for the automaton. We find it very interesting to note such a similarity despite the very different microscopic dynamics involved.

We recall once more that the theory is not complete in the sense that the avalanche occurrence frequency must be additionally supplied; it should be derived from the microscopic dynamics. To be more precise as to what a microscopic theory should yield, we note that within the framework of the master equation, the whole behavior is determined once the transition probability rate $W(H,H')$ is known; in the present case this reduces to the function $Q_0(H)$ [cf. Eq. (3.3)]. Here we have presented the consistency between the results of the theory and the automaton, but we have not shown a derivation of $Q_0(H)$ in terms of the specific rules of evolution of the automaton. It is of interest to note that, although it is not possible to perform a direct measurement of $Q_0(H)$, it can be extracted from the knowledge of the stationary solution Eq. (3.7) from which

$$Q_0(H)=\Omega\frac{d}{dH}\ln\frac{P_s(H)}{P_0}. \quad (5.1)$$

All the quantities on the right-hand side of this equation are measurable and we can numerically calculate the logarithm and the derivative with respect to H. Figure 10 shows the result. Even though we use approximately 5×10^4 avalanches, the statistics become very poor for large H. Moreover, $Q_0(H)$ seems to be diverging at a value near 0.36 while the highest conceivable possible value H_{max} is equal to 0.75. The latter corresponds to the situation where all sites in the lattice have energy $h_{ij}=0.75$: The following addition of energy ϵ will trigger, with probability 1, a large avalanche that will leave the system with zero total energy. Thus, since the probability to find the system with energies greater than 0.36 becomes very small, it is possible that we may be facing a percolation type of transition, such that $H_{max}<0.75$.

As a final comment, we point out the importance of the time distributions $T(t)$ and $S(t)$, related to the probabili-

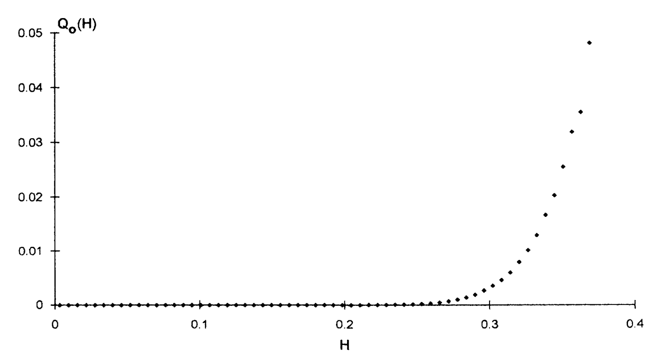

FIG. 10. Avalanche initiation probability rate $Q_0(H)$ as a function of energy. The statistics are poor for high values of H; see Fig. 8 and the text.

ties of occurrence of an event at time t given that one occurred at $t=0$, [cf. Eqs. (3.26) and (3.31)]. Besides their obvious value in seismological studies, these distributions have the quality that they are the easiest quantities to measure. As we showed in Sec. III, these distributions can be derived from the master equation and therefore are related to the details of the process. Thus the stochastic nature of these processes and the probabilistic predictions given by the present approach are fully contained in these distributions.

ACKNOWLEDGMENTS

This work was supported by Grants UNAM-DGAPA-IN-104193 and 106694.

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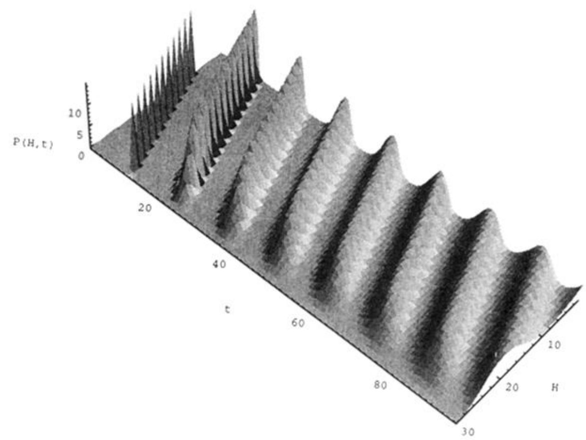

FIG. 9. Combined plot of $P(H,t|0)$ vs t and H.